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Oscillation Criteria for a Nonlinear Hyperbolic Equation Boundary Value Problem

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Abstract—In this paper, we study a class of delay hyperbolic equations boundary value problems, and obtain sufficient conditions for the oscillation of solutions of the equation (E) with two kinds of boundary conditions.

Keywords—Oscillation, Hyperbolic equation, Distributed deviating arguments.

1. INTRODUCTION

Recently, people began to pay attention to the oscillatory behavior of hyperbolic equations with delay, and have had some results; we can refer to the works [1–9] and their references. Those works, however, only considered the case of discrete delay. The corresponding theory is as yet not well developed. Up till now, there are few results for hyperbolic equations with continuous distributed deviating arguments. Liu and Fu [10] and Wang [11] have considered a class of hyperbolic equations with continuous distributed deviating arguments, respectively. In this paper, we will consider the following nonlinear hyperbolic equations:

$$\begin{aligned}\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u+\sum_{i=1}^{n}p_i(t)u(x,\tau_i(t))\right)\right] &= a(t)\Delta u+\sum_{j=1}^{m}a_j(t)\Delta u(x,\rho_j(t)) \\ -q(x,t)u-\int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])\,d\sigma(\xi), &\qquad (x,t)\in\Omega\times R_+\end{aligned} \tag{E}$$

and the boundary value conditions of the following types:

$$\frac{\partial u}{\partial n}+\nu(x,t)u=0, \qquad \text{on } (x,t)\in\partial\Omega\times R_+, \tag{B$_1$}$$

$$u=0, \qquad \text{on } (x,t)\in\partial\Omega\times R_+, \tag{B$_2$}$$

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where Δu is the Laplacian in R^n, $(x,t) \in \Omega \times R_+ = G, R_+ = [0,+\infty), u = u(x,t)$. $\nu(x,t) \in C(\partial\Omega \times R_+, R_+), \Omega$ is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$. n denotes the unit exterior normal vector to $\partial\Omega$.

The aim of this paper is to obtain some new oscillatory criteria for equation (E) satisfying two kinds of boundary value conditions.

We assume throughout this paper that the following Conditions (H) hold.

(H_1) $p(t)$, $a(t)$, $p_i(t)$, $a_j(t)$, $\rho_j(t) \in C(R_+, R_+)$, $\tau_i(t) \le t$, $\rho_j(t) \le t$, and $\tau_i(t)$, $\rho_j(t)$ are nondecreasing, $i = 1, 2, \dots, n$, $j = 1, 2, \dots m$, and $\lim_{t\to+\infty} \tau_i(t) = \lim_{t\to+\infty} \rho_j(t) = +\infty$.

(H_2) $q(x,t) \in C(\overline{\Omega} \times R_+, R_+)$, $q(x,t,\xi) \in C(\overline{\Omega} \times R_+ \times [a,b], R_+)$, $f(u) \in C(R,R)$ is a convex in R_+ and $-f(-u) = f(u)$.

(H_3) $g(t,\xi) \in C(R_+ \times [a,b], R)$; $g(t,\xi) \le t$, $\xi \in [a,b], g(t,\xi)$ are nondecreasing with to t, ξ, respectively, and $\lim_{t\to+\infty} \min_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty$.

(H_4) $\sigma(\xi) \in ([a,b], R)$ is nondecreasing, integral of equation (E) is a Stieltjes integral.

It is easy to see that equation (E) includes the following delay hyperbolic equation:

$$\begin{aligned} \frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u + \sum_{i=1}^{n} p_i(t)u(x,\tau_i(t))\right)\right] &= a(t)\Delta u + \sum_{j=1}^{m} a_j(t)\Delta u(x,\rho_j(t)) \\ &-q(x,t)u - \sum_{k=1}^{s} q_k(x,t)f(u[x,g_k(t)]), \qquad (x,t) \in G, \end{aligned} \tag{E$'$}$$

and we can note that the hyperbolic equations in [1–9] and in [10,11] all are special cases of equation (E$'$) and equation (E). Some of our results extend and improve some given results in [1–11].

DEFINITION 1. *A function $u \in C^2(G) \cap C^1(\overline{G})$ is called a solution of problems (E) and (B), if it satisfies (E) in the domain G along with the corresponding boundary condition.*

DEFINITION 2. *A solution $u(x,t)$ of equation (E) is called oscillatory in the domain G if for each positive number μ, there exists a point $(x_0,t_0) \in \Omega \times [\mu,+\infty)$ such that the condition $u(x_0,t_0) = 0$ holds.*

2. MAIN RESULTS

Now we let

$$Q(t) = \min_{x\in\Omega}\{q(x,t)\}, \qquad Q(t,\xi) = \min_{x\in\Omega}\{q(x,t,\xi)\}. \tag{2.1}$$

With each solution $u(x,t)$ of problems (E) and (B_1), we associate a $U(t)$ defined by

$$U(t) = \frac{\int_\Omega u(x,t)\,dx}{\int_\Omega dx}. \tag{2.2}$$

THEOREM 1. *Suppose that Condition (H) holds, and*

$$0 \le \sum_{i=1}^{n} p_i(t) \le 1 \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{1}{p(s)}\,ds = +\infty, \tag{2.3}$$

$$f(u) \ge \varepsilon u > 0, \qquad (u \ne 0, \varepsilon \text{ is a positive constant}). \tag{2.4}$$

If

$$\int_{t_0}^{+\infty}\int_a^b Q(s,\xi)\left[1 - \sum_{i=1}^{n} p_i[g(s,\xi)]\right] d\sigma(\xi)\,ds = +\infty, \tag{2.5}$$

then the every solution of equation (E) and (B_1) is oscillatory in G.

PROOF. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of Problems (E) and (B_1). Without loss of generality, we may assume that $u(x,t) \geq 0$, $(x,t) \in \Omega \times [\mu, +\infty)$ $(\mu \geq 0)$, by Condition (H_3), there exists a $t_1 > \mu$ such that $g(t,\xi) \geq \mu$, $(t,\xi) \in [t_1, +\infty) \times [a,b]$, and $\tau_i(t) \geq \mu$, $\rho_j(t) \geq \mu, t \geq t_1$, then

$$\begin{aligned} u[x, g(t,\xi)] > 0, \qquad & (x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b], \\ u(x, \tau_i(t)) > 0, \qquad u(x, \rho_j(t)) > 0, \quad & (x,t) \in \Omega \times [t_1, +\infty). \end{aligned}$$

Integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u\,dx + \sum_{i=1}^n p_i(t)\int_\Omega u(x,\tau_i(t))\,dx\right)\right] \\ &\quad = a(t)\int_\Omega \Delta u\,dx + \sum_{j=1}^m a_j(t)\int_\Omega \Delta u(x,\rho_j(t))\,dx \\ &\qquad - \int_\Omega q(x,t)u\,dx - \int_\Omega\int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])\,d\sigma(\xi)\,dx, \qquad t \geq t_1. \end{aligned} \tag{2.6}$$

Using Green's formula, we have

$$\int_\Omega \Delta u\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,d\omega = -\int_{\partial\Omega} \nu(x,t)u\,d\omega \leq 0, \tag{2.7}$$

$$\int_\Omega \Delta u(x,\rho_j(t))\,dx = \int_{\partial\Omega} \frac{\partial u(x,\rho_j(t))}{\partial n}\,d\omega = -\int_{\partial\Omega} \nu(x,\rho_j(t))u(x,\rho_j(t))\,d\omega \leq 0. \tag{2.8}$$

Using Jensen's inequality and (2.1), we have

$$\begin{aligned} \int_\Omega\int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])\,d\sigma(\xi)\,dx &= \int_a^b\int_\Omega q(x,t,\xi)f(u[x,g(t,\xi)])\,dx\,d\sigma(\xi) \\ &\geq \int_a^b Q(t,\xi)\left[\int_\Omega f(u[x,g(t,\xi)])\,dx\right]d\sigma(\xi) \\ &\geq \int_a^b Q(t,\xi)\left\{f\left(\frac{\int_\Omega u[x,g(t,\xi)]\,dx}{\int_\Omega dx}\right)\int_\Omega dx\right\}d\sigma(\xi), \qquad t \geq t_1, \end{aligned} \tag{2.9}$$

$$\int_\Omega q(x,t)u\,dx \geq Q(t)\int_\Omega u\,dx, \qquad t \geq t_1, \tag{2.10}$$

therefore, from (2.6)–(2.10), we have

$$\begin{aligned} &\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u\,dx + \sum_{i=1}^n p_i(t)\int_\Omega u(x,\tau_i(t))\,dx\right)\right] + Q(t)\int_\Omega u\,dx \\ &\qquad + \int_a^b Q(t,\xi)\left\{f\left(\frac{\int_\Omega u[x,g(t,\xi)]\,dx}{\int_\Omega dx}\right)\int_\Omega dx\right\}d\sigma(\xi) \leq 0, \end{aligned}$$

then

$$\begin{aligned} &\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) + \sum_{i=1}^n p_i(t)U(\tau_i(t))\right)\right] \\ &\qquad + Q(t)U(t) + \int_a^b Q(t,\xi)f(U[g(t,\xi)])\,d\sigma(\xi) \leq 0, \qquad t \geq t_1. \end{aligned}$$

Let

$$Z(t) = U(t) + \sum_{i=1}^{n} p_i(t) U(\tau_i(t)). \tag{2.11}$$

It is easy to obtain $Z(t) > 0$, $[p(t)Z'(t)]' < 0$ for $t \geq t_1$. Hence, $p(t)Z'(t)$ is decreasing in t, and we can prove that $Z'(t) \geq 0$ for $t \geq t_1$. In fact, suppose that $Z'(t) < 0$ for $t \geq t_1$, then there exists a $T > t_1$ such that $p(T)Z'(T) < 0$. Then $p(t)Z'(t) \leq p(T)Z'(T)$ for $t \geq T$; it follows that

$$Z(t) \leq Z(T) + p(T)Z'(T) \int_T^t \frac{1}{p(s)}\, ds;$$

therefore, $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts the fact that $Z(t) > 0$. Furthermore, from above inequality and (2.1),(2.3), we have

$$\begin{gathered}
\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) + \sum_{i=1}^{n} p_i(t) U(\tau_i(t))\right)\right] + \int_a^b Q(t,\xi) f(U[g(t,\xi)])\, d\sigma(\xi) \leq 0, \qquad t \geq t_1, \\
f(U[g(t,\xi)]) \geq \varepsilon U[g(t,\xi)], \\
U[g(t,\xi)] = Z[g(t,\xi)] - \sum_{i=1}^{n} p_i[g(t,\xi)] U(\tau_i[g(t,\xi)]), \\
[p(t)Z'(t)]' + \varepsilon \int_a^b Q(t,\xi)\left[Z[g(t,\xi)] - \sum_{i=1}^{n} p_i[g(t,\xi)] U(\tau_i[g(t,\xi)])\right] d\sigma(\xi) \leq 0.
\end{gathered} \tag{2.12}$$

Noticing $Z(t) \geq U(t)$, and $\tau_i(t), Z(t)$ are nondecreasing in t, we have

$$[p(t)Z'(t)]' + \varepsilon \int_a^b Q(t,\xi)\left[1 - \sum_{i=1}^{n} p_i[g(t,\xi)]\right] Z[g(t,\xi)]\, d\sigma(\xi) \leq 0. \tag{2.13}$$

Now we choose a constant $K > 0$ such that $Z(K) > 0$ and from (H_3), there exists a sufficiently large T such that $g(t,\xi) > K$, $t > T$, $\xi \in [a,b]$, thus, we have $Z[g(t,\xi)] \geq Z(K)$. From (2.13), we obtain

$$[p(t)Z'(t)]' + \varepsilon Z(K) \int_a^b Q(t,\xi)\left[1 - \sum_{i=1}^{n} p_i[g(t,\xi)]\right] d\sigma(\xi) \leq 0.$$

Integrating both sides of the above inequality from T to t ($t > T$), we have

$$p(t)Z'(t) - p(T)Z'(T) + \varepsilon Z(K) \int_T^t \int_a^b Q(s,\xi)\left[1 - \sum_{i=1}^{n} p_i[g(s,\xi)]\right] d\sigma(\xi)\, ds \leq 0. \tag{2.14}$$

Taking $t \to +\infty$, the above last inequality leads to a contradiction with (2.5).

The case that $u(x,t)$ is an eventually negative solution of (E),(B_1) can be proved by the analogous argument. This completes the proof of Theorem 1.

THEOREM 2. *Suppose that Condition (H) and (2.3),(2.4) hold. If there exist* $\frac{d}{dt}g(t,a)$ *and function* $\psi(t) \in C^1([t_0,+\infty), R_+)$ *such that*

$$\int_{t_0}^{+\infty} \left\{\varepsilon\psi(s) \int_a^b Q(s,\xi)\left[1 - \sum_{i=1}^{n} p_i[g(s,\xi)]\right] d\sigma(\xi) - \frac{p(s)p[g(s,a)]\psi'^2(s)}{4\psi(s)g'(s,a)}\right\} ds = +\infty, \tag{2.15}$$

then all solutions $u(x,t)$ of Problem (E) and (B_1) oscillate in G.

PROOF. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of Problem (E) and (B_1). Without loss of generality, we may assume that $u(x,t) \geq 0$, $(x,t) \in \Omega \times [\mu,+\infty)$ $(\mu \geq 0)$. Similar to the proof of Theorem 1, we have

$$Z'(t) \geq 0 \quad \text{and} \quad [p(t)Z'(t)]' \leq 0 \tag{2.16}$$

and

$$[p(t)Z'(t)]' + \varepsilon \int_a^b Q(t,\xi) \left[1 - \sum_{i=1}^n p_i[g(t,\xi)]\right] Z[g(t,\xi)]\, d\sigma(\xi) \le 0. \tag{2.17}$$

Noticing that $g(t,\xi)$ is nondecreasing in ξ, we have $g(t,a) \le g(t,\xi)$, $\xi \in [a,b]$, then

$$[p(t)Z'(t)]' + \varepsilon Z[g(t,a)] \int_a^b Q(t,\xi) \left[1 - \sum_{i=1}^n p_i[g(t,\xi)]\right] d\sigma(\xi) \le 0. \tag{2.18}$$

Let

$$W(t) = \psi(t)\frac{p(t)Z'(t)}{Z[g(t,a)]},$$

then we obviously have $W(t) > 0$, for $t > t_1$, and using the condition of the theorem, we see that there exist $Z'[g(t,a)] = \frac{dZ}{dg}\frac{dg}{dt}g(t,a)$. From $[p(t)Z'(t)]' < 0$ and $g(t,\xi) \le t$, $\xi \in [a,b]$, we have $p(t)Z'(t) \le p[g(t,a)]Z'[g(t,a)]$, thus

$$\begin{aligned}
W'(t) &= \frac{\psi'(t)p(t)Z'(t)}{Z[g(t,a)]} - \frac{\psi(t)p(t)Z'(t)Z'[g(t,a)]g'(t,a)}{Z^2[g(t,a)]} + \psi(t)\frac{[p(t)Z'(t)]'}{Z[g(t,a)]} \\
&\le \frac{\psi'(t)\,[p(t)Z'(t)]}{Z[g(t,a)]} - \frac{\psi(t)p(t)Z'^2(t)g'(t,a)}{p[g(t,a)]Z^2[g(t,a)]} + \psi(t)\frac{[p(t)Z'(t)]'}{Z[g(t,a)]} \\
&= \psi(t)\frac{[p(t)Z'(t)]'}{Z[g(t,a)]} + \frac{p(t)p[g(t,a)]\psi'^2(t)}{4\psi(t)g'(t,a)} \\
&\quad - \left[\sqrt{\frac{p(t)\psi(t)g'(t,a)}{p[g(t,a)]}}\frac{Z'(t)}{Z[g(t,a)]} - \sqrt{\frac{p(t)p[g(t,a)]}{2\psi(t)g'(t,a)}}\psi'(t)\right]^2 \\
&\le \psi(t)\frac{[p(t)Z'(t)]'}{Z[g(t,a)]} + \frac{p(t)p[g(t,a)]\psi'^2(t)}{4\psi(t)g'(t,a)},
\end{aligned}$$

then, it follows from (2.18) that

$$W'(t) \le -\left\{\varepsilon\psi(t)\int_a^b Q(t,\xi)\left[1 - \sum_{i=1}^n p_i[g(t,\xi)]\right] d\sigma(\xi) - \frac{p(t)p[g(t,a)]\psi'^2(t)}{4\psi(t)g'(t,a)}\right\}.$$

Integrating both sides of the above last inequality from t_1 to t ($t > t_1$), we get

$$W(t) \le W(t_1) - \int_{t_1}^t \left\{\varepsilon\psi(s)\int_a^b Q(s,\xi)\left[1 - \sum_{i=1}^n p_i[g(s,\xi)]\right] d\sigma(\xi) - \frac{p(s)p[g(s,a)]\psi'^2(s)}{4\psi(s)g'(s,a)}\right\} ds.$$

Taking $t \to +\infty$, the above last inequality leads to a contradiction with (2.15).

The case that $u(x,t)$ is an eventually negative solution of (E) and (B_1) can be proved by the analogous argument. This completes the proof of Theorem 2.

COROLLARY 1. *Suppose that Condition (H) holds. If the following differential inequality*

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(U(t) + \sum_{i=1}^n p_i(t)U(\tau_i(t))\right)\right] + Q(t)U(t) + \int_a^b Q(t,\xi)f\left(U[g(t,\xi)]\right) d\sigma(\xi) \le 0 \tag{2.19}$$

has no eventually positive solution, then the every solution of equations (E) and (B_1) is oscillatory in G.

In Theorem 2, choosing $\psi(t) \equiv 1$, and canceling the condition of the existing $\frac{d}{dt}g(t,a)$, then Theorem 2 is not distinct from Theorem 1.

Now we consider the oscillation of (E) and (B_2).

With each solution $u(x,t)$ of Problems (E) and (B_2), associate a $V(t)$ defined by

$$V(t) = \frac{\int_\Omega u(x,t)\Phi(x)\,dx}{\int_\Omega \Phi(x)\,dx}. \tag{2.20}$$

For the following Dirichlet problem in the domain Ω,

$$\Delta u + \alpha u = 0, \qquad \text{in } (x,t) \in \partial\Omega \times R_+, \tag{2.21}$$
$$u = 0, \qquad \text{on } (x,t) \in \partial\Omega \times R_+, \tag{2.22}$$

in which α is a constant.

It is well known [12] that the smallest eigenvalue α_1 of problem (2.21) is positive and the corresponding eigenfunction $\Phi(x) \geq 0$, for $x \in \Omega$.

THEOREM 3. *Suppose that Condition (H) and (2.3),(2.4) hold, if there exists function $\varphi(t) \in C^1([t_0,+\infty), R_+)$ such that*

$$\int_{t_0}^{+\infty} \left\{ \alpha_1 \varphi(s) \sum_{j=1}^{m} a_j(s) \left[1 - \sum_{i=1}^{n} p_i\left(\rho_j(s)\right)\right] - \frac{[\varphi'(s)]^2}{4\varphi(s)} \right\} ds = +\infty, \tag{2.23}$$

then the every solution of equations (E) and (B_2) is oscillatory in G.

PROOF. Let $u(x,t)$ be a positive solution of Problems (E) and (B_2) in $\Omega \times [\mu,+\infty)\,(\mu \geq 0)$. Similar to the proof of Theorem 1, there exists a $t_1 > \mu$, such that $u[x,g(t,\xi)] > 0$, $(x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]$, $u(x,\tau_i(t)) > 0$, $u(x,\rho_j(t)) > 0$, $(x,t) \in \Omega \times [t_1,+\infty)$.

Multiplying both sides of equation (E) by $\Phi(x)$, and integrating with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u\Phi(x)\,dx + \sum_{i=1}^{n} p_i(t)\int_\Omega u(x,\tau_i(t))\Phi(x)\,dx\right)\right] &= a(t)\int_\Omega \Delta u\Phi(x)\,dx \\ + \sum_{j=1}^{m} a_j(t)\int_\Omega \Delta u(x,\rho_j(t))\Phi(x)\,dx - \int_\Omega q(x,t)u\Phi(x)\,dx \\ - \int_\Omega \int_a^b q(x,t,\xi) f(u[x,g(t,\xi)])\Phi(x)\,d\sigma(\xi)\,dx, \qquad t \geq t_1. \end{aligned} \tag{2.24}$$

Using Green's formula, we have

$$\begin{aligned} \int_\Omega \Delta u\Phi(x)\,dx &= \int_{\partial\Omega}\left(\Phi(x)\frac{\partial u}{\partial n} - u\frac{\partial\Phi(x)}{\partial n}\right) d\omega \\ &\quad + \int_\Omega u\Delta\Phi(x)\,dx = -\alpha_1 \int_\Omega u\Phi(x)\,dx, \qquad t \geq t_1 \end{aligned} \tag{2.25}$$

$$\int_\Omega \Delta u(x,\rho_j(t))\Phi(x)\,dx = -\alpha_1 \int_\Omega u(x,\rho_j(t))\Phi(x)\,dx, \qquad t \geq t_1 \tag{2.26}$$

in which α_1 is the smallest eigenvalue of the problem.

Using Jensen's inequality, we have

$$\begin{aligned} \int_\Omega \int_a^b q(x,t,\xi) f(u[x,g(t,\xi)])\Phi(x)\,d\sigma(\xi)\,dx &= \int_a^b \int_\Omega q(x,t,\xi) f(u[x,g(t,\xi)])\Phi(x)\,dx\,d\sigma(\xi) \\ &\geq \int_a^b Q(t,\xi)\left[\int_\Omega f(u[x,g(t,\xi)])\Phi(x)\,dx\right] d\sigma(\xi) \\ &\geq \int_a^b Q(t,\xi)\left\{ f\left(\frac{\int_\Omega u[x,g(t,\xi)]\Phi(x)\,dx}{\int_\Omega \Phi(x)\,dx}\right)\int_\Omega \Phi(x)\,dx\right\} d\sigma(\xi), \qquad t \geq t_1, \end{aligned} \tag{2.27}$$

$$\int_\Omega q(x,t)u\Phi(x)\,dx \geq Q(t)\int_\Omega u\Phi(x)\,dx, \qquad t \geq t_1, \tag{2.28}$$

therefore, from (2.23)–(2.28), we have

$$\begin{aligned}
&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u\Phi(x)\,dx + \sum_{i=1}^n p_i(t)\int_\Omega u(x,\tau_i(t))\Phi(x)\,dx\right)\right] + \alpha_1 a(t)\int_\Omega u\Phi(x)\,dx \\
&+ \alpha_1 \sum_{j=1}^m a_j(t)\int_\Omega u(x,\rho_j(t))\Phi(x)\,dx + Q(t)\int_\Omega u\Phi(x)\,dx \\
&+ \int_a^b Q(t,\xi)\left\{f\left(\frac{\int_\Omega u[x,g(t,\xi)]\Phi(x)\,dx}{\int_\Omega \Phi(x)\,dx}\right)\int_\Omega \Phi(x)\,dx\right\}d\sigma(\xi) \leq 0, \qquad t \geq t_1,
\end{aligned}$$

then

$$\begin{aligned}
\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^n p_i(t)V(\tau_i(t))\right)\right] &+ \alpha_1\left(a(t)V(t) + \sum_{j=1}^m a_j(t)V(\rho_j(t))\right) \\
&+ Q(t)V(t) + \int_a^b Q(t,\xi)f(V[g(t,\xi)])\,d\sigma(\xi) \leq 0,
\end{aligned}$$

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^n p_i(t)V(\tau_i(t))\right)\right] + \alpha_1\sum_{j=1}^m a_j(t)V(\rho_j(t)) \leq 0. \tag{2.29}$$

Let

$$Y(t) = V(t) + \sum_{i=1}^n p_i(t)V(\tau_i(t)).$$

It is easy to obtain $Y(t) > 0$, $[p(t)Y'(t)]' < 0$, for $t \geq t_1$. Hence, $p(t)Y'(t)$ is decreasing in t, and we can also prove that $Y'(t) > 0$.

Noticing that $\tau_i(t), \rho_j(t)$ are nondecreasing, and $Y(t) \geq V(t)$, we have

$$[p(t)Y'(t)]' + \alpha_1\sum_{j=1}^m a_j(t)\left[1 - \sum_{i=1}^n p_i(\rho_j(t))\right]Y(\rho_j(t)) \leq 0. \tag{2.30}$$

Letting $\rho(a) = \min \rho_j(t)$, from (2.29) and letting $Y(t)$ be increasing in t, we have

$$[p(t)Y'(t)]' + \alpha_1 Y(\rho(a))\sum_{j=1}^m a_j(t)\left[1 - \sum_{i=1}^n p_i(\rho_j(t))\right] \leq 0. \tag{2.31}$$

Let

$$G(t) = \varphi(t)\frac{p(t)Y'(t)}{Y(\rho(a))}.$$

The remainder of the proof is similar to that of Theorem 1, and so we omit it.

The case that $u(x,t)$ is an eventually negative solution of (E) and (B_2) can be proved by the analogous argument. This completes the proof of Theorem 2.

COROLLARY 2. *Suppose that Condition (H) holds. If the following differential inequality*

$$\begin{aligned}
\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^n p_i(t)V(\tau_i(t))\right)\right] &+ \alpha_1\left(a(t)V(t) + \sum_{j=1}^m a_j(t)V(\rho_j(t))\right) \\
&+ Q(t)V(t) + \int_a^b Q(t,\xi)f(V[g(t,\xi)])\,d\sigma(\xi) \leq 0
\end{aligned}$$

has no eventually positive solution, then the every solution of equations (E) and (B_2) is oscillatory in G.

In Theorem 3, choosing $\varphi(t) \equiv 1$, we have the following.

COROLLARY 3. *Suppose that Condition (H) and (2.3)–(2.5) hold, if*

$$\int_{t_0}^{+\infty} \alpha_1 \sum_{j=1}^{m} a_j(s) \left[1 - \sum_{i=1}^{n} p_i(\rho_j(t))\right] ds = +\infty,$$

then the every solution of equations (E) and (B_2) is oscillatory in G.

The following theorem and lemma can be proved analogously.

THEOREM 4. *Suppose that the conditions of Theorem 1 hold. Then every solution of equations (E) and (B_2) is oscillatory in G.*

COROLLARY 4. *Suppose that the conditions of Corollary 1 hold. Then every solution of equations (E) and (B_2) is oscillatory in G.*

To conclude this paper, we consider an example.

$$\begin{gathered}\frac{\partial}{\partial t}\left[4\frac{\partial}{\partial t}\left(u + \frac{1}{8}u(x, t-\pi)\right)\right] = 3\frac{\partial^2 u}{\partial x^2} - 2u + 3\frac{\partial^2}{\partial x^2}u\left(x, t - \frac{3}{2}\pi\right) \\ -6\int_{-\pi/2}^{-\pi/4} u(x, t+2\xi)\, d\xi, (x,t) \in (0,\pi) \times R_+ \end{gathered} \tag{2.32}$$

and the boundary value condition of the following type:

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \qquad t \geq 0,$$

in which $n = m = 1$, $p(t) = 4$, $p_i(t) = 1/8$, $a(t) = 3$, $a_1(t) = 3$, $q(x,t) = 2$, $q(x,t,\xi) = 6$,

$$\tau_1(t) = \pi, \qquad \rho_1(t) = \frac{3\pi}{2}, \qquad g(t,\xi) = t + 2\xi.$$

Choosing $\psi(t) = \sqrt{t}$, $\varepsilon = 1$, it is easily proved that the conditions of Theorem 2 are satisfied. Therefore, every solution of equation (E) is oscillatory in $(0,\pi) \times (0,+\infty)$. In fact, $u(x,t) = \sin x \cos t$ is such a solution.

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